On the convergence of Fourier series in every arrangement of the terms

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1. In our earlier paper [2] the following theorem was proved:

Theorem A. *If $\{\varrho(n)\}$ is any sequence of positive numbers for which*

$$\varrho(n)=o(\log\log n),\ ^{1})$$

then there exists a square integrable function on $(0, 2\pi)$ whose Fourier series

$$\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1}$$

is such that

$$\sum_{n=1}^{\infty}(a_n^2+b_n^2)\varrho(n)<\infty,$$

and which can be rearranged into an everywhere divergent series

$$\sum_{k=1}^{\infty}(a_{n(k)}\cos n(k)x+b_{n(k)}\sin n(k)x).$$

In this paper we are going to sharpen this result slightly by making use of an observation of Tandori [1]. Our theorem reads as follows:

Theorem 1. *Let $\eta(<1)$ be a positive number. There exists a sequence of numbers $a_1, b_1, \ldots, a_n, b_n, \ldots$ such that*

$$\sum_n(a_n^2+b_n^2)\log\log n\,(\log\log\log n)^{1-\eta}<\infty, \tag{2}$$

and the Fourier series (1) can be rearranged into an almost everywhere divergent series.

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[1]) In this paper log means logarithm with base 4 but, of course, this is not essential in our considerations.

As to the partial sums of rearranged Fourier series we have the following estimate:

Theorem 2. *If $\eta(<1)$ is a positive number, then there exists a square integrable function whose Fourier series*

$$\sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

can be rearranged in such a way that the partial sums $S_l(x)$ of the rearranged series

$$\sum_{k=1}^{\infty}(A_{n(k)} \cos n(k)x + B_{n(k)} \sin n(k)x)$$

satisfy the relation

$$\overline{\lim_{l\to\infty}} \frac{|S_l(x)|}{\sqrt{\log\log l(\log\log\log l)^{1-\eta}}} > 0$$

almost everywhere.

We remark that in the assertions of Theorem 1 and Theorem 2 the words "almost everywhere" can be replaced by "everywhere". This refinement needs the same technique as can be found in [2]. (See there Lemma 4.)

It seems to be very probable that the analogous results hold in the case of Walsh's orthogonal system, but we do not treat it here.

In the sequel we shall use the following notation as an abbreviation: $\lambda(n) = \log\log n(\log\log\log n)^{1-\eta}$ if $n \geqq 17$, and $\lambda(n)=1$ if $1 \leqq n \leqq 16$, where $\eta(<1)$ is a given positive number.

2. The proof of Theorem 1 is based on the same idea as was worked out in our cited paper [2] to prove Theorem A, except for that paper's Lemma 3, which will be improved in a simple way. Let us recall here this lemma:

Lemma 1.[2]) *Let α be a natural number, and let $\varepsilon(<\pi/2)$ be a positive number. Then there exist mutually disjoint*[3]) *trigonometric polynomials $R_k^{(i)}(x)$ and simple sets $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i;\ i=1, 2, \ldots)$ with the following properties:*

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[2]) Here we give the original lemma with a little modification with respect to the frequencies occurring in $R_k^{(i)}(x)$ $(k=1, 2, \ldots, 3^i)$. Originally, we stated that they were contained between f_{i-1} and f_i $(i=1, 2, \ldots)$. Our new assertion concerning the occurring frequencies is essentially included in the proof of the lemma in question.

[3]) For a function $a_n \cos nx + b_n \sin nx$ $(\not\equiv 0)$ we call n its frequency. Two trigonometric polynomials are said to be disjoint if they have no terms of the same frequency.

(i) *the frequencies occurring in* $R_k^{(i)}(x)$ $(k=1, 2, \ldots, 3^i)$ *are contained between* $f_{\alpha+i-1}$ *and* $f_{\alpha+i}$ *where* $f_j=(C_1/\varepsilon)^j 4^{4^j}$ $(j=1, 2, \ldots)$;[4])

$$\text{(ii)} \qquad \int_{-\pi}^{\pi}\left(\sum_{k=1}^{3^i} R_k^{(i)}(x)\right)^2 dx \leqq C_2 \quad \textit{for} \quad i=1, 2, \ldots;$$

(iii) *the sets* $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$ *corresponding to the same value of i are disjoint., the set*

$$F_i=\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]-\bigcup_{k=1}^{3^i} E_k^{(i)}$$

consists of at most $f_{\alpha+i}$ *disjoint intervals, the lengths of which are at least* $1/f_{\alpha+i}$, *and*

$$\text{(4)} \qquad \operatorname{mes}(F_i) \leqq \varepsilon\left(1-\frac{1}{2^i}\right); \text{[5])}$$

(iv) *for any natural number i, the trigonometric polynomials* $R_k^{(j)}(x)$ *with* $k=1, 2, \ldots, 3^j$; $j=1, 2, \ldots, i$ *can be arranged into a sequence*

$$U_1^{(i)}(x), U_2^{(i)}(x), \ldots, U_{J_i}^{(i)}(x), \quad \textit{where} \quad J_i=3+3^2+\cdots+3^i;$$

such that

$$\text{(5)} \qquad \sum_{l=1}^{\mu_k^{(i)}} U_l^{(i)}(x) \geqq \frac{i}{8} \quad \textit{for every} \quad x \in E_k^{(i)}$$

with $\mu_k^{(i)}(\leqq J_i)$ *not depending on the particular point x in* $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$.

The only new observation — but an essential one — is the fact that the trigonometric polynomials $R_k^{(j)}(x)$ corresponding to the different values of j have to be considered with different "weights". More precisely, we give the following stronger form of Lemma 1:

Lemma 2. *Let* $i(\geqq 4)$ *and* α *be natural numbers,* $1<\alpha \leqq \sqrt{i}$, *and let* ε *be a real number,* $4^{-4^\alpha}<\varepsilon<\pi/2$. *Furthermore, let* $R_k^{(j)}(x)$ $(k=1, 2, \ldots, 3^j;\ j=1, 2, \ldots, i)$ *and* $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$ *be, respectively, the disjoint trigonometric polynomials and the disjoint simple sets in the sense of Lemma 1; and let*

$$\text{(6)} \qquad D_j=\begin{cases} \dfrac{1}{\alpha \log \alpha} & \textit{if} \quad 1 \leqq j \leqq \alpha, \\[2ex] \dfrac{1}{j \log j} & \textit{if} \quad \alpha<j \leqq i. \end{cases}$$

[4]) In the following $C_1, C_2, \ldots$ will denote positive constants.

[5]) mes (F) denotes the Lebesgue measure of the set F.

Then the coefficients $a_n = a_n^{(j)}$, $b_n = b_n^{(j)}$ *defined by*

$$\sum_{k=1}^{3^j} D_j R_k^{(j)}(x) = \sum_{n=f_{\alpha+j-1}+1}^{f_{\alpha+j}} (a_n \cos nx + b_n \sin nx) \qquad (j = 1, 2, \ldots, i)$$

fulfil the inequality

$$\sum_{n=f_\alpha+1}^{f_{\alpha+i}} (a_n^2 + b_n^2)\lambda(n) \leqq \frac{C_3}{(\log\alpha)^\eta}. \tag{7}$$

Denoting by

$$V_1^{(i)}(x), V_2^{(i)}(x), \ldots, V_{J_i}^{(i)}(x)$$

the arrangement of the trigonometric polynomials $D_j R_k^{(j)}(x)$ $(k = 1, 2, \ldots, 3^j;$ $j = 1, 2, \ldots, i)$ *made in the same way as in Lemma 1, we have*

$$\sum_{l=1}^{\mu_k^{(i)}} V_l^{(i)}(x) \geqq C_4 \log \frac{\log i}{\log\alpha} \tag{8}$$

for every $x \in E_k^{(i)}$ $(k = 1, 2, \ldots, 3^i)$.

Lemma 2 can be proved by nearly the same argument as Lemma 1. For this reason we give only a sketch of the proof.

First of all, on the basis of the condition on ε, we find

$$\lambda(f_{\alpha+j}) \leqq \begin{cases} \lambda(f_{2\alpha}) \leqq C_5\alpha(\log\alpha)^{1-\eta} & \text{if } 1 \leqq j \leqq \alpha, \\ \lambda(f_{2j}) \leqq C_5\, j(\log j)^{1-\eta} & \text{if } \alpha < j \leqq i. \end{cases}$$

From (ii) and (6), by a simple calculation, we obtain

$$\sum_{n=f_\alpha+1}^{f_{\alpha+i}} (a_n^2 + b_n^2)\lambda(n) \leqq \sum_{j=1}^{i} \lambda(f_{\alpha+j}) D_j^2 \int_{-\pi}^{\pi} \left(\sum_{k=1}^{3^j} R_k^{(j)}(x)\right)^2 dx \leqq$$

$$= C_2\left\{\sum_{j=1}^{\alpha} + \sum_{j=\alpha+1}^{i}\right\}\lambda(f_{\alpha+j}) D_j^2 \leqq C_2 C_5 \left\{\frac{1}{(\log\alpha)^{1+\eta}} + \sum_{j=\alpha+1}^{i} \frac{1}{j(\log j)^{1+\eta}}\right\} \leqq \frac{C_3}{(\log\alpha)^\eta}, \text{[6]}$$

which is the assertion (7).

[6] The last estimate follows from

$$\sum_{j=\alpha+1}^{i} \frac{1}{j(\log j)^{1+\eta}} \leqq C_6 \int_{\alpha}^{i} \frac{dx}{x(\log x)^{1+\eta}} \leqq \frac{C_6}{(\log\alpha)^\eta} \qquad (\eta > 0).$$

We shall also make use of another inequality that says

$$\sum_{j=\alpha+1}^{i} \frac{1}{j\log j} \geqq C_7 \int_{\alpha}^{i} \frac{dx}{x\log x} \geqq C_7\,(\log\log i - \log\log\alpha).$$

As to (8), by applying (5) and (6), we get in the same way as in the proof of Lemma 1 that

$$\sum_{l=1}^{\mu_k^{(i)}} V_l(x) \geqq \frac{1}{8}\sum_{j=1}^{i} D_j = \frac{1}{8}\left\{\frac{1}{\log\alpha}+\sum_{j=\alpha+1}^{i}\frac{1}{j\log j}\right\} \geqq C_4 \log\frac{\log i}{\log\alpha} \quad (x\in E_k^{(i)}),$$

which concludes the proof of Lemma 2.

3. Proof of Theorem 1. Set $\varepsilon_m = 1/m$ $(m=1, 2, \ldots)$, and define the increasing sequences of natural numbers $\{\alpha_m\}$ and $\{i_m\}$ by recurrance with respect to m as follows: for $m=1$ put $\alpha_1=2$ and $i_1=4$, furthermore, the numbers α_{m+1} and i_{m+1} are chosen so large that the following conditions are satisfied for $m=1, 2, \ldots$:

$$(9)\qquad \begin{gathered} f_{\alpha_m+i_m} = (C_1 m)^{\alpha_m+i_m}4^{4^{\alpha_m+i_m}} < (C_1(m+1))^{\alpha_{m+1}}4^{4^{\alpha_{m+1}}} = f_{\alpha_{m+1}}, \\ \frac{1}{(\log\alpha_{m+1})^{\eta}} \leqq \frac{1}{(m+1)^2} \quad\text{and}\quad i_{m+1}=\alpha_{m+1}^2. \end{gathered}$$

It is clear that these choices are possible.

Then apply subsequently Lemma 2 with $\varepsilon=\varepsilon_m$, $\alpha=\alpha_m$ and $i=i_m$ $(m=1, 2, \ldots)$. We obtain the trigonometric polynomials $V_l^{(i_m)}(x)$ $(l=1, 2, \ldots, J_{i_m};\ m=1, 2, \ldots)$ satisfying (7) and (8), and denote by $T_m(x)$ the sum of all $V_l^{(i_m)}\left(x-\frac{m\pi}{2}\right)$ corresponding to the same value of i_m. It is obvious that

$$T_m(x) = \sum_{n=f_{\alpha_m}+1}^{f_{\alpha_m+i_m}} (a_n\cos nx + b_n \sin nx) \qquad (m=1, 2, \ldots).$$

Now consider the series

$$(10)\qquad \sum_{m=1}^{\infty} T_m(x).$$

By virtue of (9) the trigonometric polynomials $T_m(x)$ and $T_{m'}(x)$ do not overlap for $m\neq m'$. Therefore, writing every $T_m(x)$ in (10) in extenso, we represent (10) in the form of trigonometric series

$$(11)\qquad \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx),$$

where the coefficients a_n and b_n not occurring in any $T_m(x)$ equal 0.

Taking into consideration (7) and (9) we obtain that the inequality (2) holds.

Now write down the mutually disjoint trigonometric polynomials $V_l^{(i_m)}\left(x-\frac{m\pi}{2}\right)$ ($l=1, 2, \ldots, J_{i_m}$; $m=1, 2, \ldots$) in this order:

$$V_1^{(i_1)}\left(x-\frac{\pi}{2}\right),\ V_2^{(i_1)}\left(x-\frac{\pi}{2}\right), \ldots, V_{J_{i_1}}^{(i_1)}\left(x-\frac{\pi}{2}\right); \ldots$$

$$\ldots; V_1^{(i_m)}\left(x-\frac{m\pi}{2}\right),\ V_2^{(i_m)}\left(x-\frac{m\pi}{2}\right), \ldots, V_{J_{i_m}}^{(i_m)}\left(x-\frac{m\pi}{2}\right); \ldots$$

and label the occurring frequencies, in this order, by the subscript $n(k)$ ($k=1, 2, \ldots$). It is clear that the series

$$\sum_{k=1}^{\infty}(a_{n(k)}\cos n(k)x+b_{n(k)}\sin n(k)x) \tag{12}$$

is a well determined arrangement of the non-vanishing terms of (11).

On account of (8) and (9), for every $m(\geqq 1)$ we find that

$$\max_{f_{\alpha_m}<l\leqq f_{\alpha_m}+i_m}\left\{\sum_{k=f_{\alpha_m}+1}^{l}(a_{n(k)}\cos n(k)x+b_{n(k)}\sin n(k)\right\}\geqq C_4\log 2$$

holds in $\left(\frac{m\pi}{2}-\frac{\pi}{4}, \frac{m\pi}{2}+\frac{\pi}{4}\right)$ except on a set F_m of measure less than $1/m$. We can easily see that almost every point of $(-\pi; \pi)$ is not contained in infinitely many F_m.[7]) Thus the series (12) is divergent almost everywhere, and the proof of Theorem 1 is complete.

4. Proof of Theorem 2. Starting with Theorem 1 it is possible to deduce Theorem 2. For this purpose we define the sequences $\{\varepsilon_m\}$, $\{\alpha_m\}$ and $\{i_m\}$ of numbers exactly as in the proof of Theorem 1, i.e. so that the conditions (9) are satisfied. We consider the trigonometric polynomials $D_j R_k^{(j)}(x)$ ($k=1, 2, \ldots, 3^j$; $j=1, 2, \ldots, i_m$) defined by Lemma 2 for $m=1, 2, \ldots$. Denote by

$$W_1^{(i_m)}(x),\ W_2^{(i_m)}(x), \ldots, W_{J_{i_m}}^{(i_m)}(x)$$

the arrangement of the trigonometric polynomials $\sqrt{\lambda(f_{\alpha_m+j})}\,D_j R_k^{(j)}(x)$ ($k=1, 2, \ldots, 3^j$; $j=1, 2, \ldots, i_m$) made in the same way as previously described in Lemma 1 concerning $R_k^{(j)}(x)$. It is clear that

$$U_m(x)=\sum_{l=1}^{J_{i_m}}W_l^{(i_m)}\left(x-\frac{m\pi}{2}\right)=\sum_{n=f_{\alpha_m}+1}^{f_{\alpha_m}+i_m}(A_n\cos nx+B_n\sin nx),$$

[7]) We consider the sets F on the whole line of real numbers modulo 2π.

where $A_n=\sqrt{\lambda(f_{\alpha_m+j})}\,a_n$ and $B_n=\sqrt{\lambda(f_{\alpha_m+j})}\,b_n$ if $f_{\alpha_m+j-1}<n\leqq f_{\alpha_m+j}$ $(j=1,2,\dots,i_m;$ $m=1,2,\dots)$, and the coefficients a_n, b_n are defined in Theorem 1.

Now we form the series

$$\sum_{m=1}^{\infty} U_m(x). \tag{13}$$

Since two polynomials $U_m(x)$ with different subscript are disjoint owing to (9), we obtain the trigonometric series

$$\sum_{n=1}^{\infty}(A_n\cos nx+B_n\sin nx) \tag{14}$$

by writing out the terms of each trigonometric polynomial in (13). The series (14) is the Fourier series of a square integrable function. Indeed, by (2), we have

$$\sum_{n=1}^{\infty}(A_n^2+B_n^2)=\sum_{m=1}^{\infty}\int_{-\pi}^{\pi}U_m^2(x)\,dx=$$

$$=\sum_{m=1}^{\infty}\sum_{j=1}^{i_m}\lambda(f_{\alpha_m+j})D_j^2\int_{-\pi}^{\pi}\left(\sum_{k=1}^{3^j}R_k^{(j)}\left(x-\frac{m\pi}{2}\right)\right)^2dx\leqq$$

$$\leqq C_8\sum_{m=1}^{\infty}\sum_{n=f_{\alpha_m}+1}^{f_{\alpha_m+i_m}}(a_n^2+b_n^2)\lambda(n)<\infty.$$

Finally, let us rearrange the terms of the series (14) as we did in (11). In this way we obtain

$$\sum_{k=1}^{\infty}(A_{n(k)}\cos n(k)x+B_{n(k)}\sin n(k)x).$$

Denote by $S_l(x)$ the lth partial sum of this series. We are going to show that

$$\max_{f_{\alpha_m}<l\leqq f_{\alpha_m+i_m}}\frac{S_l(x)-S_{f_{\alpha_m}}(x)}{\sqrt{\lambda(f_{\alpha_m+i_m})}}$$

does not tend to zero almost everywhere (although $\alpha_m\to\infty$). To achieve this aim, let us consider the trigonometric polynomials $V_l^{(i_m)}(x)$ $(l=1,2,\dots,J_{i_m})$ and the simple sets $E_k^{(i_m)}$ $(k=1,2,\dots,3^{i_m})$ in the sense of Lemma 2. For every $x\in E_k^{(i_m)}$ $(k=1,2,\dots,3^{i_m})$ we have

$$\sum_{l=1}^{\mu_k^{(i_m)}}V_l^{(i_m)}(x)=\sum_{j=1}^{i_m}\Big({\sum_k}^{*}D_jR_k^{(j)}(x)\Big)=\sum_{j=1}^{i_m}\frac{1}{\sqrt{\lambda(f_{\alpha_m+j})}}\Big({\sum_k}^{*}D_j\sqrt{\lambda(f_{\alpha_m+j})}\,R_k^{(j)}(x)\Big),$$

where the sum ${\sum_k}^{*}$ is extended over every integer value of k $(1\leqq k\leqq 3^j)$ for which the trigonometric polynomial $D_jR_k^{(j)}(x)$ is equal to some $V_l^{(i_m)}(x)$ occurring on the left-hand side sum of this inequality ($j=1, 2, \ldots, i_m$). Performing an Abel transform we find that

$$\sum_{l=1}^{\mu_k^{(i_m)}} V_l^{(i_m)}(x) =$$

$$(15) \qquad = \sum_{l=1}^{i_m-1} \left(\frac{1}{\sqrt{\lambda(f_{\alpha_m+l})}} - \frac{1}{\sqrt{\lambda(f_{\alpha_m+l+1})}} \right) \sum_{j=1}^{l} \Big(\sum_k^* D_j \sqrt{\lambda(f_{\alpha_m+j})}\, R_k^{(j)}(x) \Big) +$$

$$+ \frac{1}{\sqrt{\lambda(f_{\alpha_m+i_m})}} \sum_{j=1}^{i_m} \Big(\sum_k^* D_j \sqrt{\lambda(f_{\alpha_m+j})}\, R_k^{(j)}(x) \Big).$$

We notice that the last term on the right-hand side of (15) equals

$$\frac{1}{\sqrt{\lambda(f_{\alpha_m+i_m})}} \sum_{l=1}^{\mu_k^{(i_m)}} W_l^{(i_m)}(x).$$

We assert that the series on the right-hand side of (15) converges to zero almost everywhere (as $m \to \infty$). By a simple calculation we obtain

$$\sum_{l=1}^{i_m-1} \left(\frac{1}{\sqrt{\lambda(f_{\alpha_m+l})}} - \frac{1}{\sqrt{\lambda(f_{\alpha_m+l+1})}} \right) \int_{-\pi}^{\pi} \left| \sum_{j=1}^{l} \Big(\sum_k^* D_j \sqrt{\lambda(f_{\alpha_m+j})}\, R_k^{(j)}(x) \Big) \right| dx \leqq$$

$$\leqq \sqrt{2\pi} \sum_{l=1}^{i_m-1} \left(\frac{1}{\sqrt{\lambda(f_{\alpha_m+l})}} - \frac{1}{\sqrt{\lambda(f_{\alpha_m+l+1})}} \right) \left\{ \sum_{j=1}^{l} D_j^2 \lambda(f_{\alpha_m+j}) \int_{-\pi}^{\pi} \Big(\sum_k^* R_k^{(j)}(x) \Big)^2 dx \right\}^{1/2} \leqq$$

$$\leqq C_8 \sum_{l=\alpha_m+1}^{\alpha_m+i_m-1} \left(\frac{1}{\sqrt{\lambda(f_l)}} - \frac{1}{\sqrt{\lambda(f_{l+1})}} \right) \left\{ \sum_{n=f_{\alpha_m}+1}^{f_{\alpha_m+i_m}} (a_n^2 + b_n^2) \lambda(n) \right\}^{1/2},$$

where we took into consideration that the trigonometric polynomials $R_k^{(j)}(x)$ ($k=1, 2, \ldots, 3^j$; $j=1, 2, \ldots, i_m$) are mutually disjoint. On account of (2), by Beppo Levi's theorem this implies our above assertion. Denote by H the set of measure zero, on which the series on the right-hand side of (15) does not tend to zero.

By virtue of (8) and (9) we get that

$$\max_{f_{\alpha_m} < l \leqq f_{\alpha_m+i_m}} \frac{S_l(x) - S_{f_{\alpha_m}}(x)}{\sqrt{\lambda(f_{\alpha_m+i_m})}} \geqq \frac{1}{2\sqrt{\lambda(f_{\alpha_m+i_m})}} \sum_{l=1}^{\mu_k^{(i_m)}} W_l^{(i_m)}\left(x - \frac{m\pi}{2} \right) \geqq \frac{C_4 \log 2}{2}$$

holds in $\left(\frac{m\pi}{2} - \frac{\pi}{4}, \frac{m\pi}{2} + \frac{\pi}{4} \right)$ provided $x \notin F_m \cup H$ and m is large enough, where

$$F_m = \bigcup_{k=1}^{3^{i_m}} E_k^{(i_m)},$$

and thus $\text{mes}(F_m) < \varepsilon_m = 1/m$. It is obvious that almost every point of $(-\pi, \pi)$ is not contained in infinitely many $F_m \cup H$. Taking into account the elementary fact that

$$\max_{f_{\alpha_m} < l \leqq f_{\alpha_m + i_m}} \frac{S_l(x) - S_{f_{\alpha_m}}(x)}{\sqrt{\lambda(f_{\alpha_m + i_m})}} \leqq 2 \max_{f_{\alpha_m} \leqq l \leqq f_{\alpha_m + i_m}} \frac{|S_l(x)|}{\sqrt{\lambda(l)}},$$

we can see that

$$\overline{\lim_{l \to \infty}} \frac{|S_l(x)|}{\sqrt{\lambda(l)}} \geqq \frac{C_4 \log 2}{4}$$

holds almost everywhere. This proves our statement (3), and finishes the proof of Theorem 2.

References

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(Received January 26, 1969)